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Long Cycles in 3-Connected Graphs

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Moon and Moser in 1963 conjectured that if G is a 3-connected planar graph on n vertices, then G contains a cycle of length at least $\Omega(n^{\log_3 2})$. In this paper, this conjecture is proved. In addition, the same result is proved for 3-connected graphs embeddable in the projective plane, or the torus, or the Klein bottle.

1. INTRODUCTION AND NOTATION

A graph is *Hamiltonian* if it contains a cycle using all vertices, and such a cycle is called a *Hamilton* cycle. A *planar graph* is a graph which can be embedded in the plane without crossing edges, and such an embedding is called a *plane graph*.

In 1931, Whitney [11] proved that every 4-connected triangulation of the plane contains a Hamilton cycle. In 1956, Tutte [10] proved a more general result: every 4-connected planar graph contains a Hamilton cycle. However, 3-connected planar graphs need not contain Hamilton cycles. For such examples, see Holton and McKay [7].

The *circumference* of a graph G, denoted by $circ(G)$, is the length of a longest cycle in G. In 1963, Moon and Moser [9] implicitely made the following conjecture by giving 3-connected planar graphs G with $circ(G) \leqslant 9|V(G)|^{\log_3 2}$.

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Conjecture 1.1. *If G is a 3-connected planar graph on n vertices, then $circ(G) \geqslant \Omega(n^{\log_3 2})$.*

We mention here that Grünbaum and Walther [6] made the same conjecture for a family of 3-connected cubic planar graphs.

Barnette [1] showed that every 3-connected planar graph with n vertices contains a cycle of length at least $\sqrt{\lg n}$ and Clark [3] later improved this lower bound to $e^{\sqrt{\lg n}}$. In [8], Jackson and Wormald obtained a polynomial lower bound βn^{α}, where β is some constant and $\alpha \approx 0.207$. Recently, Gao and Yu [5] improved α to 0.4 and extended the result to 3-connected graphs embeddable in the projective plane, or the torus, or the Klein bottle.

The main result of this paper is the following.

THEOREM 1.2. *Let G be a 3-connected graph with n vertices, and suppose that G is embeddable in the sphere, or the projective plane, or the torus, or the Klein bottle. Then $circ(G) \geqslant \Omega(n^{\log_3 2})$.*

Throughout this paper, we consider finite graphs with no loops or multiple edges. For a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. A graph H is a *subgraph* of G, denoted by $H \subset G$, if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. We shall use $\emptyset$ to denote the empty graph (as well as the empty set).

For $S \subset V(G)$, the subgraph of G *induced by* S, denoted by $G[S]$, is the graph whose vertex set is S and whose edge set consists of the edges in G with both incident vertices in S. For $S \subset E(G)$, the subgraph of G *induced by* S, denoted by $G[S]$, is the graph whose edge set is S and whose vertex set consists of the vertices in G incident with edges in S. Let e be an edge of G with incident vertices x and y; then we write $e = xy$, and write $G[xy]$ or $G[e]$ instead of $G[\{xy\}]$ or $G[\{e\}]$.

Let $X \subset V(G)$, or $X \subset E(G)$, or $X \subset G$; then $G - X$ denotes the graph obtained from G by deleting X and the edges of G incident with a vertex in X. If $X = \{x\}$, then we write $G - x$ instead of $G - \{x\}$.

Let $H \subset G$; then G/H denotes the graph with $V(G/H) = V(G - H) \cup \{h\}$ (where $h \notin V(G)$) and $E(G/H) = E(G - H) \cup \{hy\colon\ y \in V(G - H)$ and $yy' \in E(G)$ for some $y' \in V(H)\}$. We say that G/H is obtained from G by *contracting* H to the vertex h. If G is a graph, and $x, y \in V(G)$, then $G + xy = G$ if $xy \in E(G)$; otherwise, $G + xy$ denotes the graph obtained from G by adding the edge xy.

Let G and H be subgraphs of a graph. Then $G \cap H$ (respectively, $G \cup H$) is the graph with vertex set $V(G) \cap V(H)$ (respectively, $V(G) \cap V(H)$) and edge set $E(G) \cap E(H)$ (respectively, $E(G) \cup E(H)$). We shall use $G - H$ instead of $G - (H \cap G)$.

A *block* of a graph G is a maximal 2-connected subgraph of G. (The complete graph on two vertices is 2-connected.) Let G be a connected graph and $X \subset V(G)$, where $|X| = k$ and k is a positive integer; then X is called a *k-cut* of G if $G - X$ is not connected. If $X - \{x\}$ is a cut set of G, then x is a *cut vertex* of G.

Let G be a plane graph, a *plane subgraph* of G is a subgraph of G inheriting the embedding of G. The *faces* of G are the connected components (in topological sense) of the complement of G in the plane. The *outer* face of a plane graph G is the unbounded face; the bounded faces are *inner faces*. The boundary of the outer face is called *the outer walk* of the graph, or the *outer cycle* if it is a cycle. A cycle is a *facial cycle* in a plane graph if it bounds a face of the graph. An *open disc* (respectively, *closed disc*) in the plane is a homeomorphic image of $\{(x, y)\colon x^2 + y^2 < 1\}$ (respectively, $\{(x, y)\colon x^2 + y^2 \leqslant 1\}$).

For $u, v \in V(G)$, a $u - v$ *path* in a graph G is a path with end vertices u and v. For any path P and $x, y \in V(P)$, xPy denotes the subpath of P between x and y. Given two vertices x and y on a cycle C in a plane graph, we use xCy to denote the path in C from x to y in clockwise order.

Let H be a subgraph of a graph G. An H-*bridge* of G is a subgraph of G which either (1) is induced by an edge of $E(G) - E(H)$ with both incident vertices in H or (2) is induced by the edges in a component of $G - H$ and the edges of G from H to that component. For any H-*bridge* B of G, the *attachments of* B (on H) are the vertices in $V(B) \cap V(H)$.

Although Theorem 1.2 is stated for 3-connected graphs, we need to work with certain 2-connected graphs. The following concepts will serve this purpose.

A *circuit graph* is a pair (G, C), where G is a 2-connected plane graph and C is a facial cycle of G, such that, for any 2-cut S of G, every component of $G - S$ contains a vertex of C.

An *annulus graph* is a triple (G, C_1, C_2), where G is a 2-connected plane graph and C_1 and C_2 are facial cycles of G, such that, for any 2-cut S of G, every component of $G - S$ contains a vertex of $C_1 \cup C_2$.

The rest of the paper is organized as follows. In Section 2, we give the Moon–Moser example which shows that the exponent $\log_3 2$ in Theorem 1.2 cannot be improved. (In fact, we slightly improve their bound of $9|V(G)|^{\log_3 2}$.) In Section 3, we prove a result for weighted graphs, from which Conjecture 1.1 follows as a consequence. In Section 4, we prove Theorem 1.2 for graphs embeddable in the projective plane, or the torus, or the Klein bottle.

2. AN EXAMPLE

In this section, we use the Moon–Moser example to illustrate that the bound in Conjecture 1.1 (and hence, Theorem 1.2) is in a sense best possible.

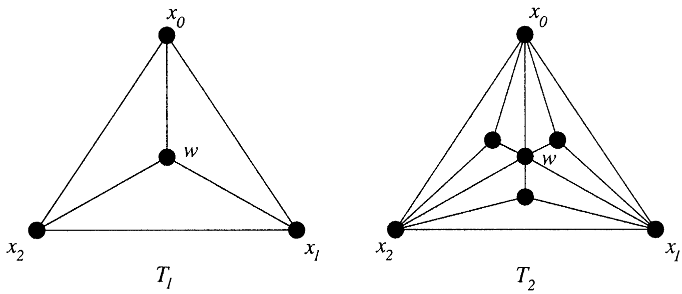

FIG. 1.

First, we define a sequence of 3-connected plane graphs $\{T_k\}$ as follows. Let T_1 be a plane graph isomorphic to K_4. Further, let $V(T_1) = \{w, x_0, x_1, x_2\}$ and let $x_0x_1x_2x_0$ be the outer cycle of T_1. Suppose that T_k is defined for some $k \geqslant 1$. Let T_{k+1} be the graph obtained from T_k as follows: in each inner face of T_k, add a new vertex and join the new vertex to the vertices of T_k incident with that face. The graphs T_1 and T_2 are shown in Fig. 1.

By the above construction, for any $k \geqslant 1$, T_k is a 3-connected plane graph with outer cycle $x_0x_1x_2x_0$. We shall show that $circ(T_k) < \frac{7}{2}n^{\log_3 2}$.

Let α_k be the length of a longest $x_1 - x_2$ path in T_k and β_k be the length of a longest $x_1 - x_2$ path in $T_k - x_0$. By the construction of T_k, for $i, j \in \{0, 1, 2\}$ and $i \neq j$, the length of a longest $x_i - x_j$ path in T_k is α_k and the length of a longest $x_i - x_j$ path in $T_k - (\{x_0, x_1, x_2\} - \{x_i, x_j\})$ is β_k.

Proposition 2.1. *For $k \geqslant 1$, $\alpha_k = 3 \cdot 2^{k-1}$ and $\beta_k = 2^k$.*

Proof. It is easy to check that $\alpha_1 = 3$ and $\beta_1 = 2$. Assume that $\alpha_k = 3 \cdot 2^{k-1}$ and $\beta_k = 2^k$. We need to show that $\alpha_{k+1} = 3 \cdot 2^k$ and $\beta_{k+1} = 2^{k+1}$.

For $i \in \{0, 1, 2\}$, let D^i denote the open disc in the plane bounded by the triangle in T_{k+1} induced by $\{w, x_0, x_1, x_2\} - \{x_i\}$. Let V^i denote the set of vertices in T_{k+1} contained in D^i, and let T^i be the plane subgraph of T_{k+1} induced by $V^i \cup (\{w, x_0, x_1, x_2\} - \{x_i\})$.

Clearly, T^i is a 3-connected plane graph isomorphic to T_k. We shall proceed with claims (a)–(d).

(a) $\alpha_{k+1} \leqslant 3 \cdot 2^k$. Let P be an $x_1 - x_2$ path in T_{k+1}. Since the outer cycle of each T^i is a triangle, $P \cap T^i$ consists of a path, or a path and an isolated vertex in $\{w, x_0, x_1, x_2\} - \{x_i\}$, or isolated vertices which are contained in $\{w, x_0, x_1, x_2\} - \{x_i\}$.

If $P \cap T^0$ is a path, then $P \subset T^0$. In this case, $|E(P)| \leqslant \alpha_k = 3 \cdot 2^{k-1} \leqslant 3 \cdot 2^k$.

If $P \cap T^1$ consists of a path and $\{w, x_0, x_2\} \subset P \cap T^1$, then either $V(P) \cap V^2 = \emptyset$ or $V(P) \cap V^0 = \emptyset$. In this case, $P \subset T^1 \cup T^j$ for some $j \in \{0, 2\}$, and

$P \cap T^j$ consists of a path and an isolated vertex. By induction, $|E(P \cap T^1)| \leqslant \alpha_k$ and $|E(P \cap T^j)| \leqslant \beta_k$. Hence, $|E(P)| \leqslant \alpha_k + \beta_k = 3 \cdot 2^{k-1} + 2^k \leqslant 3 \cdot 2^k$.

Similarly, if $P \cap T^2$ consists of a path and $\{w, x_0, x_1\} \subset P \cap T^2$, then $|E(P)| \leqslant 3 \cdot 2^k$.

Thus, we may assume that, for $i \in \{0, 1, 2\}$, $P \cap T^i$ is not a path containing $\{w, x_0, x_1, x_2\} - \{x_i\}$. Hence, for $i \in \{0, 1, 2\}$, $P \cap T^i$ consists of a single vertex, or a vertex and a path, or two isolated vertices. Therefore, by induction, $|E(P \cap T^i)| \leqslant \beta_k$, and so, $|E(P)| \leqslant 3\beta_k = 3 \cdot 2^k$. Hence, $\alpha_{k+1} \leqslant 3 \cdot 2^k$.

(b) $\alpha_{k+1} \geqslant 3 \cdot 2^k$. By induction, $T^0 - x_2$ contains an $x_1 - w$ path P_0 with $|E(P_0)| = \beta_k$, $T^2 - x_1$ contains a $w - x_0$ path P_1 with $|E(P_1)| = \beta_k$, and $T^1 - w$ contains an $x_0 - x_2$ path with $|E(P_2)| = \beta_k$. Hence, $P = P_0 \cup P_1 \cup P_2$ is an $x_1 - x_2$ path in T_{k+1} with length $3\beta_k = 3 \cdot 2^k$. Hence, $\alpha_{k+1} \geqslant 3 \cdot 2^k$.

By (a) and (b), $\alpha_{k+1} = 3 \cdot 2^k$.

(c) $\beta_{k+1} \leqslant 2^{k+1}$. Let Q be an $x_1 - x_2$ path in $T_{k+1} - x_0$. If $V(Q) \cap V^0 = \emptyset$, then $V(Q) \subset (T^1 \cup T^2) - \{x_0\}$. In this case, by induction, $|E(Q \cap T^i)| \leqslant 2^k$, and so, $|E(Q)| \leqslant 2 \cdot 2^k = 2^{k+1}$.

So assume that $Q \cap V^0 \neq \emptyset$. If $Q \subset T^0$, then $|E(Q)| \leqslant \alpha_k = 3 \cdot 2^{k-1} \leqslant 2^{k+1}$. So assume that $Q \not\subset T^0$. Since $x_0 \notin Q$, either $V(Q) \cap V^1 = \emptyset$ or $V(Q) \cap V^2 = \emptyset$. By symmetry assume that $V(Q) \cap V^1 = \emptyset$. Then $Q \cap T^2 \neq \emptyset$ is a $w - x_1$ path in $T^2 - x_0$, and $Q \cap T^0$ consists of a $w - x_2$ path in $T^0 - x_1$ and the isolated vertex x_1. Hence, $|E(Q)| = |E(Q \cap T^0)| + |E(Q \cap T^2)| \leqslant 2\beta_k = 2^{k+1}$. Thus, $\beta_{k+1} \leqslant 2^{k+1}$.

(d) $\beta_{k+1} \geqslant 2^{k+1}$. For $i \in \{1, 2\}$, $T^i - x_0$ has an $x_i - w$ path P_i of length $\beta_k = 2^k$. Hence, $P_1 \cup P_2$ is an $x_1 - x_2$ path in $T_{k+1} - x_0$ such that $|E(P_1 \cup P_2)| = 2^{k+1}$. Thus, $\beta_{k+1} \geqslant 2^{k+1}$.

By (c) and (d) $\beta_{k+1} = 2^{k+1}$. ∎

By the definition of T_{k+1}, we have $n = |V(T_{k+1})| = 4 + 3 + 3^2 + \cdots + 3^k = 3 + (3^{k+1} - 1)/2$.

Next, we show that $circ(T_{k+1}) \leqslant \alpha_k + 2\beta_k$. Let C be a longest cycle in T_{k+1}, and let T^i be defined as in the proof of Proposition 2.1. If C uses an edge of the outer cycle of T_{k+1}, then $|E(C)| \leqslant \alpha_{k+1} + 1 \leqslant \alpha_k + 2\beta_k$. If $C \cap T^i$ for some i, then by induction, $|E(C)| \leqslant \alpha_{k-1} + 2\beta_{k-1} \leqslant \alpha_k + 2\beta_k$. So assume that $C \not\subset T^i$ for any $i \in \{0, 1, 2\}$. Note that $E(C \cap T^i)$ induces a path in T^i between two vertices on the outer cycle of T^i, and at most one of these $C \cap T^i$ contains all vertices of the outer cycle of T^i. Hence, $|E(C)| \leqslant \alpha_k + 2\beta_k$. Since $n = 3 + (3^{k+1} - 1)/2$ and by a simple calculation, $|E(C)| < \frac{7}{2} n^{\log_3 2})$.

Therefore, the bound in Conjecture 1.1 is best possible.

3. PLANAR GRAPHS

In this section, we prove Conjecture 1.1. The following elementary result is needed.

LEMMA 3.1. *Let m, n, k be non-negative real numbers. Then*

(1) $m^r + n^r \geqslant (m+n)^r$ *for* $0 < r < 1$.
(2) $m^r + n^r \geqslant (m+n+k)^r$ *if* $0 \leqslant r \leqslant \log_3 2$ *and* $k = \min\{m, n, k\}$.

Proof. To prove (1), it is sufficient to show that $f(x) = x^r + (1-x)^r \geqslant 1$, where $0 < r < 1$ and $0 < x < 1$. This can be done by showing that $f(x)$ has a unique critical point $x = 1/2$ in $[0, 1]$ and $f(1/2) > 1$ (and so, the absolute minimum of $f(x)$ on $[0, 1]$ is 1).

Next, we prove (2). Without loss of generality, assume that $m \geqslant n \geqslant k$.

If $m \geqslant n \geqslant (m+n+k)/3$, then $m^r + n^r \geqslant (2/3^r)(m+n+k)^r \geqslant (m+n+k)^r$ (since $2/3^r \geqslant 1$ when $0 \leqslant r \leqslant \log_3 2$).

Hence, we may assume that $n = (\frac{1}{3} - t)(m+n+k)$, where $0 < t < 1/3$. Since $m \geqslant n > k$, $m \geqslant (m+n+k) - 2n = (\frac{1}{3} + 2t)(m+n+k)$. Thus, $m^r + n^r \geqslant [(\frac{1}{3} + 2t)^r + (\frac{1}{3} - t)^r](m+n+k)^r$. Hence, it is sufficient to show that $f(t) = (\frac{1}{3} + 2t)^r + (\frac{1}{3} - t)^r \geqslant 1$ for $0 \leqslant t \leqslant \frac{1}{3}$.

Note that $f(1/3) = 1$. Since $0 \leqslant r \leqslant \log_3 2$, $f(0) \geqslant 1$. Also, a simple calculation shows that $f(1/6) > 1$. Since $f'(t) = 2r(\frac{1}{3} + 2t)^{r-1} - r(\frac{1}{3} - t)^{r-1} = 0$ has a unique solution, $f(t) \geqslant 1$ when $0 \leqslant t \leqslant 1/3$. ∎

In order to prove Conjecture 1.1, we need to work with a class of 2-connected planar graphs which includes all 3-connected planar graphs.

DEFINITION 3.2. *Let (G, C) be a circuit graph, and let $x, y \in V(C)$. We say that (G, xCy) is a strong circuit graph if, for any 2-cut S of G, $S \cap V(yCx - \{x, y\}) \neq \emptyset$.*

We also need to work with graphs in which vertices are assigned non-negative weights. Let $\mathbb{R}^+$ denote the set of non-negative real numbers. Let G be a graph, and $w : V(G) \to \mathbb{R}^+$. For $H \subset G$, *we write* $w(H) = \Sigma_{v \in V(H)} w(v)$. Define $w(\emptyset) = 0$.

THEOREM 3.3. *Let (G, xCy) be a strong circuit graph, and let $w : V(G) \to \mathbb{R}^+$. Then G contains an $x - y$ path P such that*

$$\Sigma_{v \in V(P-y)} [w(v)]^{\log_3 2} \geqslant [w(G-y)]^{\log_3 2}.$$

Remark. In the inequality of Theorem 3.3, y is not included. This is for a technical reason which facilitates counting. Also note the symmetry between x and y (we can always re-embed G so that the clockwise direction of C becomes the counter clockwise direction).

Proof. We use induction on $|V(G)| + |E(G)|$. Since G is 2-connected and since G contains the cycle C, $|V(G)| + |E(G)| \geqslant 6$. If $|V(G)| + |E(G)| = 6$, then G is the complete graph on three vertices, and the inequality in Theorem 3.3 follows from (1) of Lemma 3.1. So assume that $|V(G) + |E(G)| > 6$. For convenience, let $r = \log_3 2$, and assume that C is the outer cycle of G. We shall proceed with claims (a)–(g).

(a) We may assume that $xy \notin E(G)$. Suppose that $xy \in E(G)$. Since (G, xCy) is a strong circuit graph, $\{x, y\}$ is not a cut set of G, and so, $xy \in E(C)$. Since G is 2-connected, we can label the cut vertices of $G - xy$ as $v_1, \ldots, v_{m-1}$ and blocks of $G - xy$ as $B_1, \ldots, B_m$ such that $B_i \cap B_{i+1} = \{v_i\}$ for $i \in \{1, 2, \ldots, m-1\}$, $B_i \cap B_j = \emptyset$ for $|i - j| \geqslant 2$ and $i, j \in \{1, \ldots, m\}$, and $v_0 = x \in B_1 - v_1$ and $v_m = y \in B_m - v_{m-1}$. See Fig. 2. In the left figure $xCy = G[xy]$ and v_0, $v_1, \ldots, v_m$ occur on C in counter clockwise order, and in the right figure $yCx = G[xy]$ and $v_0, v_1, \ldots, v_m$ occur on C in clockwise order. We view each B_i as a plane subgraph of G.

Next, we shall find a $v_{i-1} - v_i$ path P_i in B_i such that $\bigcup_{i=1}^{m} P_i$ gives the desired path P.

If $|V(B_i)| = 2$, then let $P_i = B_i$. It is easy to see that

$$\Sigma_{v \in V(P_i - v_i)} [w(v)]^r \geqslant [w(B_i - v_i)]^r.$$

So assume that $|V(B_i)| \geqslant 3$. Then $|V(B_i)| + |E(B_i)| \geqslant 6$. Let C_i denote the outer cycle of B_i.

We claim that $(B_i, v_{i-1}C_iv_i)$ is a strong circuit graph. Let S be an arbitrary 2-cut of B_i. If $B_i - S$ contains a component T with $T \cap C_i = \emptyset$, then T is also

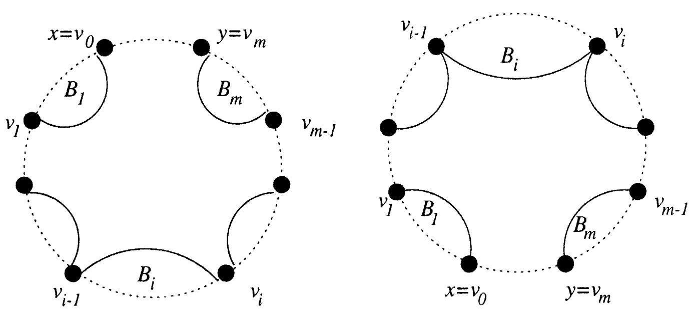

FIG. 2.

a component of $G - S$ with $T \cap C = \emptyset$, and so, (G, C) is not a circuit graph, a contradiction. Hence, any component of $B_i - S$ contains a vertex of C_i, and so, (B_i, C_i) is a circuit graph. Now assume that $S \cap (v_iC_iv_{i-1} - \{v_i, v_{i-1}\}) = \emptyset$. Then $S \subset v_{i-1}C_iv_i$. If $xCy = G[xy]$, then $G - S$ contains component T with $T \cap C = \emptyset$, a contradiction. So $yCx = G[xy]$. Then S is a 2-cut of G such that $S \cap (yCx - \{x, y\}) = \emptyset$, and so, (G, xCy) is not a strong circuit graph, a contradiction. Hence $(B_i, v_{i-1}C_iv_i)$ is a strong circuit graph.

By induction, B_i contains a $v_{i-1} - v_i$ path P_i such that

$$\Sigma_{v \in V(P_i - v_i)}[w(v)]^r \geqslant [w(B_i - v_i)]^r.$$

Now let $P = \bigcup_{i=1}^m P_i$. Then P is an $x - y$ path in G. Moreover,

$$\begin{aligned}\Sigma_{v \in V(P-y)}[w(v)]^r &= \sum_{i=1}^{m} (\Sigma_{v \in V(P_i - v_i)}[w(v)]^r) \\ &\geqslant \sum_{i=1}^{m} [w(B_i - v_i)]^r \\ &\geqslant [w(G - y)]^r.\end{aligned}$$

The first inequality follows from the previous inequalities about P_i. Since $V(G - y)$ is the disjoint union of $V(B_i - v_i)$ for $i \in \{1, \ldots, m\}$, the second inequality follows from (1) of Lemma 3.1. This completes the proof of (a).

Since (G, xCy) is a strong circuit graph, x is not a cut vertex of $G - y$. Hence, $G - y$ contains a unique block, say H, containing x. Since $xy \notin E(G)$ and since H is a block of $G - y$, $|V(H)| + |E(H)| \geqslant 6$. Let D denote the outer cycle of H.

Let $y_1, y_2, \ldots, y_m \in V(D)$ denote the attachments of $(H \cup \{y\})$-bridges of G, and assume that $x, y_1, \ldots, y_m$ occur on D in this clockwise order. See Fig. 3.

(b) For $i \in \{1, \ldots, m\}$, there are exactly two $\{y, y_i\}$-bridges of G; and for $i \in \{1, \ldots, m-1\}$, $yy_i \in E(G)$ and the $\{y, y_i\}$-bridge of G not containing x is $G[yy_i]$.

Since (G, xCy) is a strong circuit graph, for any 2-cut S of G, $S \cap (yCx - \{x, y\}) \neq \emptyset$. Hence, for $i \in \{1, \ldots, m-1\}$, $\{y, y_i\}$ is not a cut set of G, and so, G has exactly two $\{y, y_i\}$-bridges, $yy_i \in E(G)$, and the $\{y, y_i\}$-bridge of G not containing x is $G[yy_i]$. If G has at least three $\{y, y_m\}$-bridges, then $G - \{y, y_m\}$ contains a component T with $T \cap C = \emptyset$, a contradiction. Thus, (b) follows.

Let $X = yCx - yCy_m$. Let $x' = x$ if $X = \{x\}$; otherwise, let x' be the end vertex of X other than x. Let B denote the minimal subgraph of $H - y_1Dy_m$ such that $X \subset B$ and B is a union of blocks of $H - y_1Dy_m$.

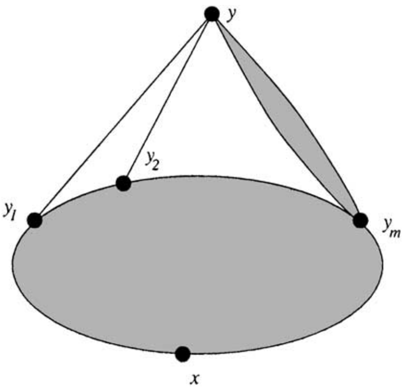

FIG. 3.

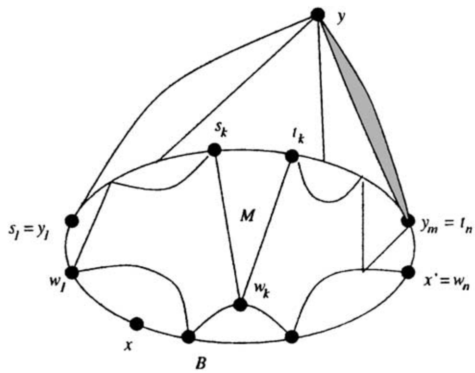

FIG. 4.

Let $w_1, \ldots, w_n \in V(B)$ be the attachments of $(B \cup y_1 D y_m)$-bridges of H. For $k \in \{1, \ldots, n\}$, define $s_k, t_k \in V(y_1 D y_m)$ as follows: y_1, s_k, t_k, y_m occur on D in this clockwise order, $\{s_k, w_k\}$ and $\{t_k, w_k\}$ are contained in $(B \cup y_1 D y_m)$-bridges of H, and subject to these conditions, $s_k D t_k$ is maximal.

We can choose the notation of $w_1, \ldots, w_n$ so that $y_1, s_1, t_1, s_2, t_2, \ldots, s_k, t_k, s_{k+1}, t_{k+1}, \ldots, s_n, t_n, y_m$ occur on D in this clockwise order. Then $y_1 = s_1$, $y_m = t_n$, and $w_n = x'$. See Fig. 4.

(c) There is some $k \in \{1, \ldots, n\}$ such that B contains an $x - w_k$ path P_B and

$$\Sigma_{v \in V(P_B)} [w(v)]^r \geqslant [w(B)]^r.$$

Let $Y = G - B$ and $B^* = G/Y$, and let y^* denote the unique vertex in $B^* - B$.

We may assume that $B \neq \{x\}$. Otherwise, $V(B^*) = \{x, y^*\}$, $E(B^*) = \{xy^*\}$, $n = 1$ and $w_1 = x = x'$. Let $P_B = \{x\}$. Clearly, P_B is an $x - w_1$ path in B such that

$$\Sigma_{v \in V(P_B)}[w(v)]^r \geqslant [w(B)]^r.$$

Hence, $|V(B^*)| + |E(B^*)| \geqslant 6$. Let C^* denote the outer cycle of B^*, where $E(C^*) = E(w_n D w_1) \cup \{w_1 y^*, y^* w_n\}$.

We claim that (B^*, xC^*y^*) is a strong circuit graph. Suppose that S is an arbitrary 2-cut of B^*. By the construction of B^*, if $y^* \in S$, then $S - y^* \subset X - x \subset y^* C^* x - \{y^*, x\}$ and the vertex in $S - y^*$ is a cut vertex of B, and so, each component of $B^* - S$ contains a vertex of C^*. Hence, we may assume that $y^* \notin S$. If T is a component of $B^* - S$, then $T \cap C^* \neq \emptyset$; otherwise, T is a component of $G - S$ with $T \cap C = \emptyset$, a contradiction. Thus, since $y^* \notin S$, $S \cap V(y^* C^* x - \{y^*, x\}) \neq \emptyset$. Hence (B^*, xC^*y^*) is a strong circuit graph.

Let $w^* : V(B^*) \to \mathbb{R}^+$ be defined as follows: $w^*(v) = w(v)$ if $v \in V(B)$, and $w^*(y^*)$ is an arbitrary non-negative integer. By induction, B^* contains an $x - y^*$ path P^* such that

$$\Sigma_{v \in V(P^* - y^*)}[w^*(v)]^r \geqslant [w^*(B^* - y^*)]^r.$$

Let $P_B = P^* - y^*$. Then P_B is a path in B from x to w_k for some $k \in \{1, \ldots, n\}$. Since $B^* - y^* = B$, and by the definition of w^*, we have

$$\Sigma_{v \in V(P_B)}[w(v)]^r \geqslant [w(B)]^r.$$

This completes (c).

Let M be the union of $s_k D t_k$ and those $(B \cup y_1 D y_m)$-bridges of H whose attachments are contained in $s_k D t_k \cup \{w_k\}$, where k is given in (c). See Fig. 4. Let $L = \emptyset$ if $s_k = y_1$; otherwise, let L denote the component of $H - (B \cup M)$ containing y_1. Let $R = G - (B \cup L \cup M \cup \{y\})$ (possibly $R = \emptyset$). Note that $V(G - y)$ is the disjoint union of $V(B)$, $V(L)$, $V(R)$, and $V(M - w_k)$.

(d) We may assume that $w(B) > \min\{w(L), w(R)\}$. Suppose that $w(B) \leqslant \min\{w(L), w(R)\}$. Let $L^* = G[V(L \cup M \cup B)]/B$, and let x^* denote the unique vertex of $L^* - (L \cup M)$. Thus x^* is the result of the contraction of B. Let $R^* = G[V(R) \cup \{y, t_k\}]$. We shall find an $x^* - t_k$ path P_L in L^* and a $t_k - y$ path P_R in R^*, and use P_L and P_R to construct the desired path P.

(d1) L^* contains an $x^* - t_k$ path P_L such that

$$\Sigma_{v \in V(P_L - \{x^*, t_k\})}[w(v)]^r \geqslant [w((L \cup M) - \{w_k, t_k\})]^r.$$

First, assume that $t_k = y_1$. Since (G, xCy) is a strong circuit graph, $L = \emptyset$, $t_k w_k \in E(G)$, $M = G[t_k w_k]$, $V(L^*) = \{x^*, t_k\}$, and $E(L^*) = \{x^* t_k\}$. Let $P_L = L^*$. Clearly,

$$\Sigma_{v \in V(P_L - \{x^*, t_k\})}[w(v)]^r = [w(\emptyset)]^r \geqslant [w((L \cup M) - \{w_k, t_k\})]^r.$$

So assume that $t_k \neq y_1$. Then L^* is a 2-connected plane graph with at least three vertices. Hence, $|V(L^*)| + |E(L^*)| \geqslant 6$. Let C_L denote the outer cycle of L^* such that $w_1 D t_k$ is the clockwise segment of C_L from $x^* = w_1$ to t_k.

Next, we show that $(L^*, t_k C_L x^*)$ is a strong circuit graph. Let S be an arbitrary 2-cut of L^*. First, assume that $L^* - S$ has a component T with $T \cap C_L = \emptyset$. Then $x^* \in S$; otherwise, T is also a component of $G - S$ with $T \cap C = \emptyset$, a contradiction. Thus, by the construction of L^*, there is some w_i, where $1 \leqslant i \leqslant k$, such that $S' = (S - \{x^*\}) \cup \{w_i\}$ is a 2-cut of G, and T is a component of $G - S'$ with $T \cap C = \emptyset$, a contradiction. Hence, (L^*, C_L) is a circuit graph. Now, assume that $S \cap (x^* C_L t_k - \{x^*, t_k\}) = \emptyset$. Then $S \subset t_k C_L x^*$. Thus, $G - S$ contains a component U with $U \cap (t_k C_L x^* - \{t_k, x^*\}) \neq \emptyset$. Hence, $w_k \neq x'$; otherwise, $t_k = y_m$ and $t_k C_L x^* - \{t_k, x^*\} = \emptyset$, a contradiction. Let $S' = S$ if $x^* \notin S$; otherwise, let $S' = (S - \{x^*\}) \cup \{w_k\}$. Thus, U is a component of $G - S'$ with $U \cap C = \emptyset$, a contradiction. Hence, $(L^*, t_k C_L x^*)$ is a strong circuit graph.

Let $w^* : V(L^*) \to \mathbb{R}^+$ be defined as follows: $w^*(v) = w(v)$ for $v \in L^* - x^*$, and $w^*(x^*) = 0$. By induction, L^* contains an $x^* - t_k$ path P_L such that

$$\Sigma_{v \in V(P_L - t_k)}[w^*(v)]^r \geqslant [w^*(L^* - t_k)]^r.$$

By the definition of w^*, we have

$$\Sigma_{v \in V(P_L - \{x^*, t_k\})}[w(v)]^r \geqslant [w((L \cup M) - \{w_k, t_k\})]^r.$$

This proves (d1).

(d2) R^* contains a $t_k - y$ path P_R such that

$$\Sigma_{v \in V(P_R - y)}[w(v)]^r \geqslant [w(R \cup \{t_k\})]^r.$$

If $|V(R^*)| = 2$, then $t_k = y_m$, $t_k y \in E(G)$, and the $\{y, y_m\}$-bridge of G not containing x is $G[t_k y]$. In this case, $R = \emptyset$. Let $P_R = R^*$. Clearly,

$$\Sigma_{v \in V(P_R - y)}[w(v)]^r \geqslant [w(R^* - y)]^r = [w(R \cup \{t_k\})]^r.$$

So assume that $|V(R^*) \geqslant 3$. Hence, $|V(R^* + t_k y)| + |E(R^* + t_k y)| \geqslant 3$. Without loss of generality, assume that $R^* + t_k y$ is a plane graph with outer cycle C_R such that $\{t_k y\} \cup E(y C y_m) \subset E(C_R)$, and $E(t_k C_R y) = \{t_k y\}$.

Next we show that $(R^* + t_k y, t_k C_k y)$ is a strong circuit graph. Suppose that S is an arbitrary 2-cut of $R^* + t_k y$. If $(R^* + t_k y) - S$ has a component T with

$T \cap C_R = \emptyset$, then T is a component of $G - S$ with $T \cap C = \emptyset$, a contradiction. Hence, $(R^* + t_k y, C_R)$ is a circuit graph, and so, $S \subset C_R$. Since $E(t_k C_R y) = \{t_k y\}$, $S \cap (yC_R t_k - \{y, t_k\}) \neq \emptyset$. Thus, $(R^* + t_k y, t_k C_k y)$ is a strong circuit graph.

By induction, R^* contains a $t_k - y$ path P_R such that

$$\Sigma_{v \in V(P_R - y)}[w(v)]^r \geqslant [w(R^* - y)]^r = [w(R \cup \{t_k\})]^r.$$

Note that we can always select P_R so that $E(P_R) \neq \{t_k y\}$. Hence, P_R is a path in R^*. This proves (d2).

Finally, we find the desired path P as follows. Without loss of generality, assume that the edge of P_L incident with x^* is $x^* v$, and v is incident with w_l, where $1 \leqslant l \leqslant k$. Let Q be a path in B from x to w_l, and let $P = ((P_L - x^*) \cup Q \cup P_R) + vw_l$. Then P is an $x - y$ path in G such that

$$\begin{aligned}\Sigma_{v \in V(P-y)}[w(v)]^r &\geqslant \Sigma_{v \in V(P_L - \{x^*, t_k\})}[w(v)]^r + \Sigma_{v \in V(P_R - y)}[w(v)]^r \\ &\geqslant [w((L \cup M) - \{w_k, t_k\})]^r + [w(R \cup \{t_k\})]^r \\ &\geqslant [w(G - y)]^r.\end{aligned}$$

Here, the first inequality is obvious, and the second inequality follows from (d1) and (d2). By the assumption that $w(B) \leqslant \min\{w(L), w(R)\}$ and since $V(G - y)$ is the disjoint union of $V((L \cup M) - \{w_k, t_k\})$, $V(B)$, and $V(R \cup \{t_k\})$, the third inequality follows from (2) of Lemma 3.1. This completes the proof of (d).

(e) We may assume that $w(L) < w(R)$. Otherwise, assume that $w(L) \geqslant w(R)$. Let $L^* = G[V(L) \cup \{y, s_k\}]$. We shall extend P_B in (c) to the desired path P by finding an $s_k - y$ path P_L in L^* and a $w_k - s_k$ path P_M in M.

(el) L^* contains an $s_k - y$ path P_L such that

$$\Sigma_{v \in V(P_L - \{s_k, y\})}[w(v)]^r \geqslant [w(L)]^r.$$

If $y_1 = s_k$, then $L = \emptyset$ and $L^* = G[yy_1]$. In this case, let $P_L = L^*$. Clearly,

$$\Sigma_{v \in V(P_L - \{s_k, y\})}[w(v)]^r = [w(L^* - \{s_k, y\})]^r = [w(\emptyset)]^r = [w(L)]^r.$$

So assume that $y_1 \neq s_k$. Then $L^* + ys_k$ is a 2-connected graph with at least three vertices. Hence, $|V(L^* + ys_k)| + |E(L^* + ys_k)| \geqslant 6$. Without loss of generality, we may assume that $L^* + ys_k$ is embedded in the plane with outer cycle C_L such that $y_1 C_L s_k = y_1 y s_k$ (and hence, $E(yC_L s_k) = \{ys_k\}$).

Next we show that $(L^* + ys_k, yC_L s_k)$ is a strong circuit graph. Let S be an arbitrary 2-cut of $L^* + ys_k$. If $(L^* + ys_k) - S$ has a component T with $T \cap C_L = \emptyset$, then T is a component of $G - S$ with $T \cap C = \emptyset$, a contradiction. Hence, $(L^* + ys_k, C_L)$ is a circuit graph, and so, $S \subset C_L$. Since $E(yC_L s_k) = ys_k$, $(L^* + ys_k, yC_L s_k)$ is a strong circuit graph.

Let $w^* : V(L^* + ys_k) \to \mathbb{R}^+$ be defined as follows: $w^*(v) = w(v)$ for $v \in L^* - s_k$, and $w^*(s_k) = 0$. By induction and by the definition of w^*, $L^* + ys_k$ contains an $s_k - y$ path P_L such that

$$\begin{aligned}\Sigma_{v\in V(P_L-\{s_k,y\})}[w(v)]^r &= \Sigma_{v\in V(P_L-y)}[w^*(v)]^r\\ &\geqslant [w^*(L^*-y)]^r\\ &= [w(L)]^r.\end{aligned}$$

Note that we can always select P_L so that $E(P_L) \neq \{ys_k\}$. Hence, P_L is a path in L^*. This proves (el).

(e2) M contains a path P_M from w_k to s_k such that

$$\Sigma_{v\in V(P_M-w_k)}[w(v)]^r \geqslant [w(M-w_k)]^r.$$

If $s_k = t_k$, then $w_ks_k \in E(G)$ and $M = G[w_ks_k]$. In this case, let $P_M = M$. Clearly,

$$\Sigma_{v\in V(P_M-w_k)}[w(v)]^r \geqslant [w(M-w_k)]^r.$$

So assume that $s_k \neq t_k$. Then $M + w_ks_k$ is a 2-connected graph with at least three vertices. Hence, $|V(M + w_ks_k)| + |E(M + w_ks_k)| \geqslant 6$. Without loss of generality, assume that $M + w_ks_k$ is embedded in the plane with outer cycle C_M such that $E(s_kDt_k) \cup \{w_ks_k\} \subset C_M$, and $E(w_kC_Ms_k) = \{w_ks_k\}$.

Next we show that $(M + w_ks_k, w_kC_Ms_k)$ is a strong circuit graph. Let S be an arbitrary 2-cut of $M + w_ks_k$. If $(M + t_ks_k) - S$ has a component T with $T \cap C_M = \emptyset$, then T is a component of $G - S$ with $T \cap C = \emptyset$, a contradiction. Thus, $(M + w_ks_k, C_M)$ is a circuit graph, and so, $S \subset C_M$. Since $E(w_kC_Ms_k) = w_ks_k$, $(M + w_ks_k, w_kC_Ms_k)$ is a strong circuit graph.

By induction, $M + w_ks_k$ contains a $w_k - s_k$ path P_M such that

$$\Sigma_{v\in V(P_M-w_k)}[w(v)]^r \geqslant [w(M-w_k)]^r.$$

Note that we can always select P_M so that $E(P_M) \neq \{w_ks_k\}$. Hence, P_M is a path in M. This proves (e2).

Now let $P = P_B \cap P_M \cup P_L$. Then P is an $x - y$ path in G. Moreover,

$$\begin{aligned}\Sigma_{v\in V(P-y)}[w(v)]^r &= \Sigma_{v\in V(P_B)}[w(v)]^r + \Sigma_{v\in V(P_M-w_k)}[w(v)]^r + \Sigma_{v\in V(P_L-\{s_k,y\})}[w(v)]^r\\ &\geqslant [w(B)]^r + [w(M-w_k)]^r + [w(L)]^r\\ &\geqslant [w(B\cup L\cup R)]^r + [w(M-w_k)]^r\\ &\geqslant [w(G-y)]^r.\end{aligned}$$

Here, the first inequality follows from (c), (el), and (e2). By (d) and the assumption that $w(L) \geqslant w(R)$, the second inequality follows from (2) of

Lemma 3.1. The third inequality follows from (1) of Lemma 3.1. This proves (e).

By (e), $R \neq \emptyset$, and so $t_k \neq y_m$. We shall extend P_B in (c) to the desired path P by finding a $t_k - y$ path P_R and a $w_k - t_k$ path P_M. The argument is similar to that for (e). Since it is not too long and is not completely symmetric to (e), we provide the details. Let $R^* = G[V(R) \cup \{y, t_k\}]$.

(f) R^* contains a $t_k - y$ path P_R such that

$$\Sigma_{v \in V(P_R - \{t_k, y\})}[w(v)]^r \geqslant [w(R)]^r.$$

If $|V(R^*)| = 2$, then $R = \emptyset$, $t_k = y_m$, $yy_m \in E(G)$, and the $\{y, y_m\}$-bridge of G not containing x is $G[yy_m]$. In this case, let $P_R = R^*$. Clearly,

$$\Sigma_{v \in V(P_R - \{t_k, y\})}[w(v)]^r = [w(\emptyset)]^r = [w(R)]^r.$$

So assume that $|V(R^*)| \geqslant 3$. Then, $|V(R^* + t_k y)| + |E(R^* + t_k y| \geqslant 6$. Without loss of generality, assume that $R^* + t_k y$ is embedded in the plane with outer cycle C_R such that $\{t_k y\} \cup E(yCy_m) \subset E(C_R)$ and $E(t_k C_R y) = \{t_k y\}$.

Now we show that $(R^* + t_k y, t_k C_R y)$ is a strong circuit graph. Let S be an arbitrary 2-cut of $R^* + t_k y$. If $(R^* + t_k y) - S$ contains a component T with $T \cap C_R = \emptyset$, then T is a component of $G - S$ with $T \cap C = \emptyset$, a contradiction. Hence, $(R^* + t_k y, C_R)$ is a circuit graph, and so, $S \subset C_R$. Since $E(t_k C_R y) = \{t_k y\}$, $(R^* + t_k y, t_k C_R y)$ is a strong circuit graph.

Let $w^* : V(R^*) \to \mathbb{R}^+$ be defined as follows: $w^*(v) = w(v)$ for $v \in V(R^* - t_k)$, and $w^*(t_k) = 0$. By induction and by the definition of $w^*, R^* + t_k y$ contains a $t_k - y$ path P_R such that

$$\begin{aligned}\Sigma_{v \in V(P_R - \{t_k, y\})}[w(v)]^r &= \Sigma_{v \in V(P_R - y)}[w^*(v)]^r \\ &\geqslant [w^*(R^* - y)]^r \\ &= [w(R)]^r.\end{aligned}$$

Note that we can always select P_R so that $E(P_R) \neq \{t_k y\}$. Hence, $P_R \subset R^*$. This completes (f).

(g) M contains a $w_k - t_k$ path P_M such that

$$\Sigma_{v \in V(P_M - w_k)}[w(v)]^r \geqslant [w(M - w_k)]^r.$$

If $s_k = t_k$, then $t_k w_k \in E(G)$ and $M = G[t_k w_k]$. In this case, let $P_M = M$. Clearly,

$$\Sigma_{v \in V(P_M - w_k)}[w(v)]^r \geqslant [w(M - w_k)]^r.$$

So assume that $s_k \neq t_k$. Then $M + t_k w_k$ is a 2-connected graph with at least three vertices. Hence, $|V(M + t_k w_k)| + |E(M + t_k w_k)| \geqslant 6$. Without loss of generality, assume that $M + t_k w_k$ is embedded in the plane with outer cycle C_M such that $E(s_k D t_k) \cup \{t_k w_k\} \subset E(C_M)$ and $E(t_k C_M w_K) = \{t_k w_k\}$.

Next, we show that $(M + t_k w_k, t_k C_M w_k)$ is a strong circuit graph. Let S be an arbitrary 2-cut of $M + t_k w_k$. If $(M + t_k w_k) - S$ has a component T with $T \cap C_M = \emptyset$, then T is a component of $G - S$ with $T \cap C = \emptyset$, a contradiction. Recall that, by (e), $t_k \neq y_m$, and so, $w_k \neq x'$. Hence, $(M + t_k w_k, C_M)$ is a circuit graph, and so, $S \subset C_M$. Since $E(t_k C_M w_k) = \{t_k w_k\}$, $(M + t_k w_k, t_k C_M w_k)$ is a strong circuit graph.

By induction, $M + t_k w_k$ contains a $t_k - w_k$ path P_M such that

$$\Sigma_{v \in V(P_M - w_k)} [w(v)]^r \geqslant [w(M - w_k)]^r.$$

Note that we can always select P_M so that $E(P_M) \neq t_k w_k$. Hence, $P_M \subset M$. This proves (g).

Now let $P = P_B \cup P_M \cup P_R$. Then P is a path in G from x to y. Moreover,

$$\begin{aligned}
\Sigma_{v \in V(P-y)} [w(v)]^r &= \Sigma_{v \in V(P_B)} [w(v)]^r + \Sigma_{v \in V(P_M - w_k)} [w(v)]^r + \Sigma_{v \in V(P_R - \{t_k, y\})} [w(v)]^r \\
&\geqslant [w(B)]^r + [w(R)]^r + [w(M - w_k)]^r \\
&\geqslant [w(B \cup L \cup R)]^r + [w(M - w_k)]^r \\
&\geqslant [w(G - y)]^r.
\end{aligned}$$

Here, the first inequality follows from (c), (f), and (g). By (d) and by (e), the second inequality follows from (2) of Lemma 3.1. The third inequality follows from (1) of Lemma 3.1.

COROLLARY 3.4. *If (G, C) is a circuit graph and $e \in E(C)$. Then G contains a cycle T through e such that $|E(T) \geqslant |V(G)|^{\log_3 2}$.*

Proof. Let $e = xy$ such that $xCy = G[xy]$. Then (G, xCy) is a strong circuit graph. Let $w : V(G) \to \mathbb{R}^+$ such that $w(v) = 1$ for all $v \in V(G)$.

By Theorem 3.3, G contains an $x - y$ path P such that

$$\begin{aligned}
|V(P)|^{-1} &= \Sigma_{v \in V(P-y)} [w(v)]^{\log_3 2} \\
&\geqslant [w(G - y)]^{\log_3 2} \\
&= (|V(G)| - 1)^{\log_3 2}.
\end{aligned}$$

Hence, by (1) of Lemma 3.1,

$$|V(P)| \geqslant 1 + (|V(G)| - 1)^{\log_3 2} \geqslant |V(G)|^{\log_3 2}.$$

Thus $P + xy$ gives the desired cycle. ∎

COROLLARY 3.5. *Let G be a 3-connected planar graph, and let $e \in E(G)$. Then G contains a cycle C through e such that $|E(C)| \geqslant |V(G)|^{\log_3 2}$.*

Proof. Without loss of generality, we assume that G is embedded in the plane such that $e \in E(C)$, where C is a facial cycle of G. Then (G, C) is a circuit graph. Hence, Corollary 3.5 follows from Corollary 3.4.

4. GRAPHS ON OTHER SURFACES

In this section, we prove that Conjecture 1.1 also holds for 3-connected graphs embedded in the projective plane, or the torus, or the Klein bottle.

DEFINITION 4.1. *Given an embedding $\sigma : G \to \Pi$ of a graph G into a surface Π, the representativity of σ is defined to be the number* $\min\{|\sigma(G) \cap \Gamma| : \Gamma$ *is a non-null homotopic simple closed curve in* $\Pi\}$.

For graphs embedded in the projective plane, Fiedler *et al.* proved the following result [4, Proposition 1]. Note that in [4], a cycle in G is called a *polygon* in G.

LEMMA 4.2. *Let $\sigma : G \to \Pi$ be an embedding of a 3-connected graph G into the projective plane Π, let P_1 be a cycle in G such that $\sigma(P_1)$ is null homotopic, and let D_1 be the open disc in Π bounded by $\sigma(P_1)$. Then there is a cycle P in G such that $\sigma(P)$ is null homotopic and bounds a disc D containing D_1, and the closure of D contains $\sigma(V(G))$.*

Note that the condition about P_1 and D_1 in Lemma 4.2 holds if the representativity of σ is at least 2. Also note that the subgraph H of G contained in the closure of D is a 2-connected spanning subgraph of G, and $\sigma(H)$ can be viewed as a plane graph embedded in the closed disc bounded by D.

LEMMA 4.3. *Let G be a 3-connected graph embeddable in the projective plane. Then G contains a 2-connected spanning planar subgraph H. Moreover, H can be embedded in the plane with a facial cycle C such that (H, C) is a circuit graph.*

Proof. Let $\sigma : G \to \Pi$ be an embedding of G in the projective plane Π. If the representativity of σ is at least 2, then by the remarks following Lemma 4.2, G has a cycle C, such that the subgraph H of G contained in the closed disc in Π bounded by $\sigma(C)$ is a spanning subgraph of G. Since G is 3-connected, it is easy to see that (H, C) is a circuit graph.

So assume that the representativity of σ is at most 1. Then G is a 3-connected planar graph. Let H be a plane embedding of G and let C be facial cycle of H. Then (H, C) is a circuit graph. ∎

Now we are ready to prove Conjecture 1.1 for 3-connected graphs embeddable in the projective plane.

THEOREM 4.4. *Let G be a 3-connected graph embeddable in the projective plane. Then $circ(G) \geqslant |V(G)|^{\log_3 2}$.*

Proof. By Lemma 4.3, let H be a 2-connected plane graph with a facial cycle C such that H is a spanning subgraph of G and (H, C) is a circuit graph. By Corollary 3.4, H, and hence G, has a cycle of length at least $|V(H)|^{\log_3 2} = |V(G)|^{\log_3 2}$. ∎

For graphs embeddable in the torus or the Klein bottle, we need the following result [2, Theorems 2 and 3].

LEMMA 4.5. *Let $\sigma : G \to \Pi$ be an embedding of a 3-connected graph G in a surface Π, where Π is the torus or the Klein bottle. Suppose σ has representativity at least 1. Then there is a spanning subgraph H of G and either (1) there is a cycle C of H such that (H, C) is a circuit graph or (2) there are two cycles C_1 and C_2 of H such that (H, C_1C_2) is an annulus graph.*

THEOREM 4.6. *Let G be a 3-connected graph embeddable in the torus or the Klein bottle. Then $circ(G) \geqslant (|V(G)|/2)^{\log_3 2}$.*

Proof. Let $\sigma : G \to \Pi$ be an embedding of G into a surface Π, where Π is the torus or the Klein bottle. If the representativity of σ is 0, then G is embeddable in the projective plane, and Theorem 4.6 follows from Theorem 4.4. So we may assume that the representativity of σ is at least 1. Then by Lemma 4.5, there is a spanning subgraph H of G and either (1) there is a cycle C of H such that (H, C) is a circuit graph or (2) there are cycles C and D of H such that (H, C, D) is an annulus graph.

If there is a cycle C of H such that (H, C) is a circuit graph, then by Corollary 3.4, H, and hence, G, contains a cycle of length at least $(|V(G))|/2)^{\log_3 2}$.

So assume that there are cycles C and D of H such that (H, C, D) is an annulus graph. Without loss of generality, we may assume that C is the outer cycle of H. We consider two cases: $C \cap D = \emptyset$ and $C \cap D \neq \emptyset$.

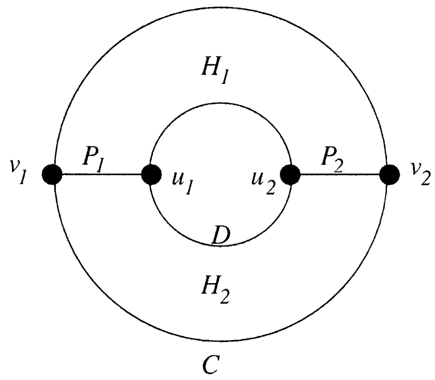

FIG. 5.

Case 1. $C \cap D = \emptyset$. Since H is 2-connected, H contains vertex disjoint paths P_1, P_2 from $v_1, v_2 \in V(C)$ to $u_1, u_2 \in V(D)$, respectively, such that, for $i \in \{1,2\}$, $(P_i - \{u_i, v_i\}) \cap (C \cup D) = \emptyset$. Let $C_1 = v_1Cv_2 \cup P_2 \cup u_1Du_2 \cup P_1$, and let $C_2 = v_2Cv_1 \cup P_1 \cup u_2Du_1 \cup P_2$. For $i \in \{1,2\}$, let H_i be the subgraph of H contained in the closed disc in the plane bounded by C_i. Then (H_i, C_i), $i \in \{1,2\}$, are circuit graphs. See Fig. 5.

Since $|V(H_1)| + |V(H_2)| > |V(H)|$, we may assume, without loss of generality, that $|V(H_1)| > |V(H)|/2$. Since (H_1, C_1) is a circuit graph, by Corollary 3.4, $circ(H_1) \geqslant |V(H_1)|^{\log_3 2}$. Hence, $circ(G) \geqslant (|V(G)|/2)^{\log_3 2}$.

Case 2. $C \cap D \neq \emptyset$. Assume that $u \in V(C \cap D)$. Let $u_1, \ldots, u_n$ be the neighbors of u in clockwise order around u such that $u_1, u_n \in V(C)$ and $u_k, u_{k+1} \in V(D)$, where $1 \leqslant k < n$. Let H' be the plane graph obtained from $H - u$ by adding two vertices u' and u'', and adding edges $u'u_i$ for $i \in \{1, \ldots, k\}$ and edges $u''u_i$ for $i \in \{k+1, \ldots, n\}$. See Fig. 6.

Since H is 2-connected, we can label the cut vertices of H' as $v_1, \ldots, v_{m-1}$ and blocks of H' as $B_1, \ldots, B_m$ such that $u, v_1, \ldots, v_{m-1}$ occur on C in this clockwise order, $B_i \cap B_{i+1} = \{v_i\}$ for $i \in \{1, \ldots, m-1\}$, $B_i \cap B_j = \emptyset$ for $i, j \in \{1, \ldots, m\}$ with $|i - j| \geqslant 2$, and $v_0 = u'' \in B_1 - v_1$ and $v_m = u' \in B_m - v_{m-1}$.

Let $W_C \subset V(H)$ (respectively, $W_D \subset V(H)$) be defined as follows: For $v \in V(H)$, $v \in W_C$ (respectively, $v \in W_D$) if, and only if, H has a 2-cut $S_v \subset V(C)$ (respectively, $S_v \subset V(D)$) such that v and $D - S_v$ (respectively, $C - S_v$) are contained in different components of $H - S_v$. Note that $W_C \cap W_D \neq \emptyset$ and $u \notin W_C \cup W_D$.

We may assume that $W_C \neq \emptyset$ and $W_D \neq \emptyset$. Otherwise, by symmetry assume that $W_D = \emptyset$. Then (H, C) is a circuit graph. Hence, by Corollary 3.4, $circ(H) \geqslant |V(H)|^{\log_3 2}$, and so, $circ(G) \geqslant (|V(G)|/2)^{\log_3 2}$.

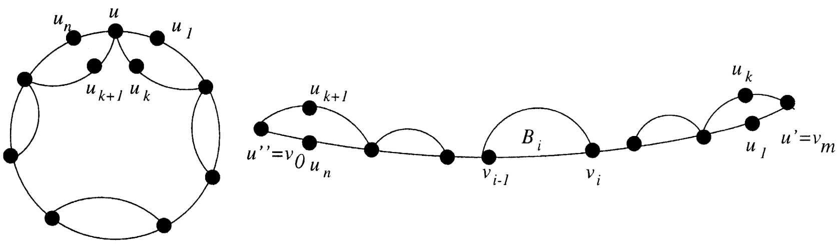

FIG. 6.

Without loss of generality, assume that $|W_D| \leqslant |W_c|$. Then, $|W_D| < |V(H)/2$ (since $u \notin W_C \cup W_D$).

Next, for $i \in \{1, \ldots, m\}$, we find a $v_{i-1} - v_i$ path P_i in B_i.

If $|V(B_i)| = 2$, then let $P_i = B_i$.

So assume that $|V(B_i)| \geqslant 3$. Then let C_i denote the outer cycle of B_i. Note that $v_{i-1}C_iv_i = v_iDv_{i-1}$ and $v_iC_iv_{i-1} = v_iCv_{i-1}$ (where we view v_0 and v_m as u). We construct a new graph B_i^* from B_i as follows.

If $W_D \cap B_i = \emptyset$, then let $B_i^* = B_i$ and let $C_i^* = C_i$.

If $W_D \cap B_i \neq \emptyset$, then $W_D \cap V(B_i)$ can be partitioned into sets $W_1^i, \ldots, W_{m_i}^i$ with the following properties: (1) for $j \in \{1, \ldots, m_i\}$, B_i has a 2-cut $S_j^i = \{s_j^i, t_j^i\} \subset V(D \cap B_i)$ and $G[W_j^i]$ is the component of $B_i - S_j^i$ not containing $C \cap B_i$, and (2) B_i has no 2-cut $S \subset V(D \cap B_i)$ such that the component of $B_i - S$ not containing $C \cap B_i$ properly contains $G[W_j^i]$. Let B_i^* denote the graph obtained from B_i by deleting $G[W_j^i]$ and adding the edges $s_j^it_j^i$ for $j \in \{1, \ldots, m_i\}$. Let C_i^* be obtained from C_i by deleting $W_j^i \cap D$ and adding edges $s_j^it_j^i$. Let the edges $s_j^it_j^i$ be added so that C_i^* is the outer cycle of B_i^* and $v_iC_i^*v_{i-1} = v_iC_iv_{i-1}$.

By the above construction, $(B_i^*, v_iC_i^*v_{i-1})$ is a strong circuit graph. Let $w : V(B_i^*) \to \mathbb{R}^+$ with $w(v) = 1$ for $v \in V(B_i^*)$. By Theorem 3.3, B_i^* contains a $v_{i-1} - v_i$ path P_i^* such that

$$\begin{aligned} |V(P_i^*)| - 1 &= \Sigma_{v \in V(P_i^* - v_i)}[w(v)]^{\log_3 2} \\ &\geqslant [w(B_i^* - v_i)]^{\log_3 2} \\ &= (|V(B_i^*)| - 1)^{\log_3 2}. \end{aligned}$$

If $s_j^it_j^i \in E(P_i^*)$, then we replace $s_j^it_j^i$ in P_i^* by a path in $G[W_j^i \cup \{s_j^it_j^i\}]$ from s_j^i to t_j^i, and let P_i denote the resulting path. Then $|V(P_i)| - 1 \geqslant |V(P_i^*)| - 1 \geqslant (|V(B_i^*)| - 1)^{\log_3 2}$.

Now let $P = \bigcup_{i=1}^{m} P_i$. Then

$$\begin{aligned}|V(P)| - 1 &= \sum_{i=1}^{m}(|V(P_i)| - 1)\\ &\geqslant \sum_{i=1}^{m}(|V(B_i^*)| - 1)^{\log_3 2}\\ &\geqslant [|V(H) - W_D|]^{\log_3 2}\\ &\geqslant (|V(G)|/2)^{\log_3 2}.\end{aligned}$$

Here, the first inequality follows from previous inequalities. Since $V(H) - W_D$ is the disjoint union of $V(B_i^* - v_i)$ for $i = 1, \ldots, m$, the second inequality follows from (1) of Lemma 3.1. The third inequality follows from the fact that $|W_D| < |V(G)|/2$.

Hence, $|V(P)| \geqslant (|V(G)|/2)^{\log_3 2} + 1$. Therefore, identifying v' and v'' in P gives the desired cycle. ∎

It is easy to see that our main result, Theorem 1.2, follows from Corollary 3.5, Theorem 4.4, and Theorem 4.6.

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